Which dynamical symmetries does the Dirac equation have?

P.R. Page* [1]

* *Theoretical Division, MS-B283, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*

Abstract. It is known that the Dirac equation has two dynamical symmetries, spin and pseudospin symmetry. Both are approximately realised in nature: spin symmetry in heavy–light mesons and pseudospin symmetry in nuclei. We prove that the spin and pseudospin symmetries are the only symmetries of their type that is possible for a parity conserving Dirac equation.

The Dirac Hamiltonian H' describing the relativistic motion of a single particle in a vector (with only a time component) and scalar potential is

$$H' = c\,\vec{\alpha}\cdot\vec{p} + \beta\,(mc^2 + V_S(\vec{r})) + V_V^0(\vec{r}) \qquad \vec{\alpha} \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad \beta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

where the usual Dirac matrices $\vec{\alpha}$ and β are written in terms of the Pauli matrices $\vec{\sigma}$, the momentum operator is $\vec{p}$, and m is the mass of the particle. If the vector potential $V_V^0(\vec{r})$ is equal to the scalar potential $V_S(\vec{r})$ plus a constant potential U, which is independent of the spatial location $\vec{r}$ of the particle from the origin, $V_V^0(\vec{r}) = V_S(\vec{r}) + U$, then the Hamiltonian is invariant under a spin symmetry [1], i.e. there exists operators $\hat{S}_i,\ i \in \{1,2,3\}$, such that H' commutes with $\hat{S}_i$, i.e. $[\,H'\,,\,\hat{S}_i\,] = 0$. If the vector potential is equal to minus the scalar potential plus a constant potential U, $V_V^0(\vec{r}) = -V_S(\vec{r}) + U$, then the Hamiltonian is invariant under a pseudospin symmetry [1]. The symmetries are called *dynamical* because they only obtain under certain assumptions about the behaviour of the potentials.

The spin symmetry is approximately realised in nature in heavy–light quark mesons systems, particularly the charm–light quark and strange–light quark systems, to the extent that the strange and charm quarks can be regarded as very heavy, so that these systems can be taken to be one–body systems [2]. The pseudospin symmetry approximately obtains in various heavy nuclei, for example the particle or hole states in double magic ^{208}Pb, which can be regarded as one–body systems [3].

[1] *E-mail:* prp@lanl.gov

Here we investigate whether the Dirac Hamiltonian has any further dynamical symmetries of the same type as the spin and pseudospin symmetries. These symmetries satisfy the following properties, which will also be assumed for the posited more general symmetries:

- $\hat{S}_i$ are Hermitean: $\hat{S}_i^\dagger = \hat{S}_i$
- $\hat{S}_i$ satisfy the $SU(2)$ algebra : $[\,\hat{S}_i\,,\,\hat{S}_j\,] = i\sum_k \epsilon_{ijk}\,\hat{S}_k$
- $\hat{S}_i$ transform like a vector: $[\,J_i\,,\,\hat{S}_j\,] = i\sum_k \epsilon_{ijk}\,\hat{S}_k$
- $[H_0, \hat{S}_i] = 0$
- $[H_{int}, \hat{S}_i] = 0$

where the kinetic part of the Hamiltonian is $H_0 = c\,\vec{\alpha}.\vec{p} + mc^2\beta$ and the potential part of the Hamiltonian is

$$H_{int} = \beta\,(V_S(\vec{r}) + \gamma_\mu V_V^\mu(\vec{r}) + \sigma_{\mu\nu} V_T^{\mu\nu}(\vec{r})) \tag{2}$$

and the total angular momentum operator is $\vec{J} = (\frac{\hbar\vec{\sigma}}{2} + \vec{r}\times\vec{p})\,\mathbf{1}$, with

$$\gamma_\mu \text{ the usual Dirac matrices, i.e.} \gamma_0 \equiv \beta, \vec{\gamma} \equiv \beta\vec{\alpha}, \text{ and } \sigma_{\mu\nu} \equiv \frac{i}{2}\,[\gamma_\mu, \gamma_\nu]\,. \tag{3}$$

The Hamiltonian $H = H_0 + H_{int}$ is the most general Hermitean local parity conserving Dirac Hamiltonian, of which the Hamiltonian H' in Eq. 1 is a special case. Parity conservation is a symmetry of QCD (which is relevant to both quark and nucleon systems). Note that the other Lorentz structures possible for the potential part of the Hamiltonian, i.e. γ_5 and $\gamma_5\gamma_\mu$, do not conserve parity.

We have proved that there are no more general symmetries than the spin and pseudospin symmetries that satisfy the properties mentioned above. We also proved that the spin and pseudospin symmetries can only commute with H if $H = H'$.

If one further restricts $H_{int} = 0$ we found a continuous one–parameter family of new symmetries $\hat{S}_i$ which satisfy the remaining properties. At the two extreme values of the parameter range one recovers the usual $\hat{S}_i$ of spin and pseudospin symmetry.

We also proved that $[\,J_i\,, H\,] = 0$ *if an only if* $V_V^i(\vec{r}) = 0,\ i \in \{1,2,3\}$, and $V_T^{\mu\nu}(\vec{r}) = 0$, and the vector and scalar potentials only depend on $|\vec{r}|$.

If one relaxes the property that $\hat{S}_i$ transforms like a vector, more symmetries have been found [1].

We are supported by the Department of Energy under contract W-7405-ENG-36.

REFERENCES

1. J.S. Bell and H. Ruegg, *Nucl. Phys.* **B98** (1975) 151.
2. P.R. Page, T. Goldman and J.N. Ginocchio, *Phys. Rev. Lett.* **86** (2001) 204.
3. J.N. Ginocchio, *these proceedings*.